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A large-N phase integral approximation for Coulomb-type systems using SO(2, 1) coherent states

Christopher C Gerry† and James B Togeas‡

† Department of Physics, St Bonaventure University, St Bonaventure, New York 14778, USA

‡ Division of Science and Mathematics, University of Minnesota, Morris, Minnesota 56267, USA

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Abstract. A phase integral approximation, based on the notion that large N is a semiclassical limit, is presented for spherically symmetric systems of the Coulomb type. Since the dynamical group SO(2, 1) is the spectrum generating group for the radial point Coulomb problem and the Casimir operators of O(N) and SO(2, 1) are coupled, we write a path integral expression of the Green function in terms of SO(2, 1) coherent states. The resulting phase integral approximation is applied to the Yukawa potential and the Coulomb plus linear r confining potential.

1. Introduction

In recent years a new technique has become popular for solving the Schrödinger equation for energy eigenvalues and eigenfunctions. We refer here to the so-called $1/N$ expansion which has proved to be of great use in quantum mechanics and field theory to explore regions beyond the reach of ordinary coupling constant perturbation theory (see Witten 1980). The usual method of implementing the large-N approximation in ordinary quantum mechanics is to consider the O(N) symmetric extension of the system of interest and employ saddle point methods, under the assumption of large N, to obtain ground-state energy levels. As pointed out by Witten (1980), the accuracy when extrapolated to $N=3$ is quite modest. However, Mlodinow and Papanicolaou (1980) have shown that the accuracy may be improved by retaining only a few terms in the perturbation series if $1/N$ is the expansion parameter. Their work is based on the use of the oscillator-type representations of SU(1, 1) and the fact that the Casimir invariants of O(N) and SU(1, 1) are coupled. They then realised the SU(1, 1) Lie algebra *à la* Holstein-Primakoff so that the natural expansion parameter is $(l+N/2)^{-1/2}$, l being the usual angular momentum quantum number.

The method described above yields impressive results for the low level energy eigenvalues for various systems such as anharmonic oscillators and even the hydrogen atom in a strong magnetic field (Bender *et al* 1982). The method is sometimes referred to as a semiclassical perturbation theory, where the expansion is made about the minimum of the effective potential in the O(N) symmetric version of the theory. However, it is also possible to understand the large-N limit as a classical limit in the sense that $\hbar \to 0$ is a classical limit (Yaffe 1982). In fact Shankar (1980), using the Bloch coherent states, has shown that the large-N limit viewed in this way leads to a

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Bohr-Sommerfeld (BS) quantisation rule for Hamiltonians expressed in terms of SU(n) generators in the symmetrised N-fold tensor product of the fundamental representation. The method is applied to the pseudospin Hamiltonian of Lipkin *et al* (1965) for $N = 30$ and 50 to obtain energy eigenvalues typically accurate to two decimal places. Following their example we have previously considered a phase integral approximation, associated with SU(1, 1) coherent states (CS) in the limit of large N, to obtain the energy eigenvalues for even power anharmonic oscillators (Gerry *et al* 1983). In this case the phase integral approximation is obtained from the path integral written over the SU(1, 1) CS (Gerry and Silverman 1982, Gerry 1982). The accuracy obtained in that calculation was comparable to the JWKB calculations for the excited states.

Now the phase integral approximation discussed above involves a realisation of the SU(1, 1) ~ SO(2, 1) generators which are of the oscillator type; in fact, they are quadratic in the boson operators a and a^+. However, another realisation of the Lie algebra exists which may be associated with the Coulomb problem (see Barut 1972). In this paper we wish to use this realisation to obtain a large-N-type phase integral approximation for Coulombic systems. It turns out that the formulation of the phase integral approximation is not as straightforward in this case as it was for the anharmonic oscillators. For one thing, the path integral must actually be written for the Green function instead of the propagator. Also it is convenient, for the purpose of calculation, to introduce a local fictitious time parameter, a trick which several authors have used to calculate the Coulomb Green function via path integrals (see Duru and Kleinhert 1979, Ho and Inomata 1982). In fact, one of the present authors has used this time modification trick to evaluate the Coulomb Green function in terms of the SO(2, 1) coherent states (Gerry 1984). (We refer to the states as SO(2, 1) coherent states as the relevant representations are those of SO(2, 1).) This will be the starting point for the present calculation.

We shall consider two potentials in this paper, both with spherical symmetry. First we consider the simple screened Coulomb potential of the Yukawa type and secondly the Coulomb potential with a term linear in r. This latter potential is sometimes referred to as the charmonium potential in view of its relevance to bound states of $q_c\bar{q}_c$ (Eichten *et al* 1978). We should stress at this point that our goal here is not to obtain energy eigenvalues of high accuracy but rather to show that one is justified in taking the large-N limit as a semiclassical approximation for the realisation and representations of SO(2, 1) relevant to the Coulomb problem.

The layout of the paper is as follows. In § 2 we briefly review the Coulomb realisation of SO(2, 1) and the coherent states for the associated representations. In § 3 we present the path integral and discuss the phase integral for the large-N limit. In § 4 the approximation is applied to the two potentials discussed above. Section 5 closes the paper with a few brief remarks. An appendix containing some useful results is included.

2. Coulomb realisation of SO(2, 1) and coherent states

We first consider the SO(2, 1) non-variance group for the Coulomb problem extended to N dimensions. By 'Coulomb' problem, however, we mean the α/r potential where

$$r = \left(\sum_{i=1}^{N} x_i^2 \right)^{1/2} \tag{2.1}$$

rather than the potential which satisfies Gauss's law in N dimensions. As pointed out by Barut and Kitagawara (1981) this may also be interpreted as an $N/3$ body problem (if N is a multiple of three). Here this distinction is immaterial. The potential α/r is now invariant under $O(N)$ transformations.

With $\boldsymbol{p}$ the N-dimensional momentum vector, the SO(2, 1) algebra is realised as

$$K_0 = \tfrac{1}{2}(rp^2 + r) \tag{2.2a}$$

$$K_1 = \tfrac{1}{2}(rp^2 - r) \tag{2.2b}$$

$$K_2 = \boldsymbol{r}\cdot\boldsymbol{p} - \tfrac{1}{2}\mathrm{i}(N-1). \tag{2.2c}$$

The Casimir operator of SO(2, 1)

$$\begin{aligned} C_2 &= K_0^2 - K_1^2 - K_2^2 \\ &= L^2 + \tfrac{3}{4}(N-1)(\tfrac{1}{3}N - 1) \end{aligned} \tag{2.3}$$

where L^2 is the Casimir operator of $O(N)$. The relevant representations are the unitary irreducible representations (UIR) $D^+(k)$, where k is the Bargmann index such that the eigenvalue of C_2 is given as $k(k-1)$. Since L^2 has the eigenvalue $l(l+N-2)$, we obtain

$$k_\pm = \tfrac{1}{2}\{1 \pm [1 + 4l(l+N-2) + 3(N-1)(\tfrac{1}{3}N-1)]^{1/2}\}. \tag{2.4}$$

For the representation to be a UIR we must have $k > 0$. For $N = 3$ we obtain $k_- = -l$ or $k_+ = l+1$. Since only k_+ satisfies this condition we hereafter drop the negative root.

The basis states of the $D^+(k)$ representation are given by $|n_r, k\rangle$ such that

$$K_0|n_r, k\rangle = (n_r + k)|n_r, k\rangle \tag{2.5}$$

where n_r is the radial quantum number. As is well known, the energy eigenvalue problem is formulated as

$$\tilde{\Omega}(E)|\tilde{\psi}\rangle = 0 \tag{2.6}$$

where

$$\begin{aligned} \tilde{\Omega}(E) &= r(H - E) \\ &= [(K_0 + K_1) - E(K_0 - K_1) - \alpha]/2\mu \end{aligned} \tag{2.7}$$

and where

$$H = (p^2/2\mu) - (\alpha/r). \tag{2.8}$$

The bound-state spectrum is obtained by transforming away the non-compact generator K_1 by the tilting operation

$$\begin{aligned} \Omega(E) &= \exp(-\mathrm{i}\theta K_2)\tilde{\Omega}(E)\exp(\mathrm{i}\theta K_2) \\ &= (-2E/\mu)^{1/2}K_0 - \alpha \end{aligned} \tag{2.9}$$

if θ is taken as $\theta = \ln[(-2E/\mu)^{1/2}]$. Then using the SO(2, 1) state $|n_r, k\rangle$ one obtains the energy eigenvalue

$$E_n = -\mu\alpha^2/2n^2 \tag{2.10}$$

where $n = n_r + k$. Obviously this reduces to the usual formula if $N = 3$.

Coherent states may be introduced on SO(2, 1), according to Perelomov (1975), as

$$|\xi, k\rangle = \exp(\alpha K_+ - \alpha^* K_-)|0, k\rangle \tag{2.11}$$

where $K_{\pm} = K_1 \pm iK_2$, $\alpha = -\frac{1}{2}e^{-i\varphi}$ and $\xi = -\tanh(\frac{1}{2}\tau)\,e^{-i\varphi}$ where τ and φ are group parameters analogous to the Euler angles. Since the SO(2, 1) states are complete as

$$\sum_{n_r=0}^{\infty} |n_r, k\rangle\langle n_r, k| = 1 \tag{2.12}$$

the coherent states of (2.11) may be expanded as

$$|\xi, k\rangle = (1-|\xi|^2)^k \sum_{n_r=0}^{\infty} \left(\frac{\Gamma(n_r+2k)}{n_r!\,\Gamma(2k)}\right)^{1/2} \xi^{n_r}|n_r, k\rangle. \tag{2.13}$$

Unity is resolved as

$$1 = \int d\mu_k(\xi)|\xi, k\rangle\langle\xi, k| \tag{2.14}$$

where

$$d\mu_k(\xi) = \frac{2k-1}{\pi}\frac{d^2\xi}{(1-|\xi|^2)^2}. \tag{2.15}$$

There also exists a reproducing kernel

$$\begin{aligned} K(\xi', \xi) &= \langle\xi', k\,|\,\xi, k\rangle \\ &= (1-|\xi'|^2)^k(1-|\xi|^2)^k(1-\xi'^*\xi)^{-2k} \end{aligned} \tag{2.16}$$

such that

$$K(\xi', \xi) = \int d\mu_k(\xi'')K(\xi', \xi'')K(\xi'', \xi). \tag{2.17}$$

These properties are of course shared by the ordinary coherent states (Glauber 1963). Finally we point out that we may define an auxiliary or physical coherent state $|\widetilde{\xi, k}\rangle$ as opposed to the group state $|\xi, k\rangle$ by the relation

$$|\widetilde{\xi, k}\rangle = \exp(i\theta K_2)|\xi, k\rangle. \tag{2.18}$$

The angle θ is to be adjusted later, but we note that equations (2.14)-(2.17) are unaltered by the replacement $|\xi, k\rangle \rightarrow |\widetilde{\xi, k}\rangle$.

3. Path integral and phase integral approximation

Consider the Hamiltonian

$$H = \frac{1}{2\mu}p^2 - \frac{\alpha}{r}U_1(r) + U_2'(r) \tag{3.1}$$

where $U_1(r)$ and $U_2'(r)$ are polynomial in r. We follow our previous work (Gerry 1984) and write the resolvent operator as

$$G(E) = -i\int_0^{\infty} \exp[-i(H-E)\tau]\,d\tau \tag{3.2}$$

and then write the Green function for a particular k sector as

$$\begin{aligned} G_k(\xi'', \xi'; E) &= \langle\widetilde{\xi'', k}|G(E)|\widetilde{\xi', k}\rangle \\ &= -i\int_0^{\infty} P_E^k(\xi'', \xi'; \tau)\,d\tau \end{aligned} \tag{3.3}$$

where

$$P_E^k(\xi'', \xi'; \tau) = \langle \widetilde{\xi'', k} | \cdot \exp[-i(H-E)\tau] | \widetilde{\xi', k} \rangle. \tag{3.4}$$

We may think of $P_E^k(\xi'', \xi'; \tau)$ as the propagator for the effective zero energy Hamiltonian $H - E$. The corresponding time evolution operator is therefore

$$\mathcal{U}(\tau) = \exp[-i\tau(H-E)]. \tag{3.5}$$

From equations (2.2) and (3.1) this may be written

$$\mathcal{U}(\tau) = \exp[-i(\tau/r)\tilde{\Omega}(E)] \tag{3.6}$$

where

$$\tilde{\Omega}(E) = (2\mu)^{-1}(K_0 + K_1) - E(K_0 - K_1) - \alpha U_1(K_0 - K_1) + U_2(K_0 - K_1) \tag{3.7}$$

and where $U_2(r) = rU_2'(r)$.

Now using techniques discussed elsewhere (Gerry and Silverman 1982, Gerry 1984), the propagator of equation (3.4) may be expressed as the path integral

$$P_E^k(\xi)'', \xi'; \tau) = \int \mathcal{D}\mu_k(\xi) \exp\left[i \int_0^\tau \left(\frac{ik}{(1-|\xi|^2)} (\xi^* \dot{\xi} - \xi \dot{\xi}^*) - \frac{1}{r(t)} \mathcal{H}(\xi, \xi^*) \right) dt \right] \tag{3.8}$$

where

$$\mathcal{H}(\xi, \xi^*) = \langle \xi, k | \Omega(E) | \xi, k \rangle \tag{3.9}$$

and

$$\begin{aligned} \Omega(E) &= \exp(-i\theta K_2)\tilde{\Omega}(E) \exp(i\theta K_2) \\ &= (2\mu)^{-1} e^{\theta}(K_0 + K_1) - e^{-\theta} E(K_0 - K_1) \\ &\quad - \alpha U_1[e^{-\theta}(K_0 - K_1)] + U_2[e^{-\theta}(K_0 - K_1)]. \end{aligned} \tag{3.10}$$

Now we introduce the fictitious time parameter $\sigma(t)$ as

$$\int^t d\sigma(t) = \int^t \frac{dt}{r(t)} \tag{3.11}$$

and write

$$P_E^k(\xi'', \xi'; \tau) = \int_0^\infty \delta\left(\tau - \int^\sigma r(s)\, ds \right) \bar{P}_E^k(\xi'', \xi'; \sigma)\, d\sigma \tag{3.12}$$

so that

$$G_k(\xi'', \xi'; E) = -i \int \bar{P}_E^k(\xi'', \xi'; \sigma)\, d\sigma \tag{3.13}$$

where

$$\bar{P}_E^k(\xi'', \xi'; \sigma) = \int \mathcal{D}\mu_k(\xi) \exp\left[i \int_0^\sigma \left(\frac{ik}{(1-|\xi|^2)} (\xi^* \xi' - \xi \xi^{*\prime}) - \mathcal{H}(\xi, \xi^*) \right) d\sigma \right]. \tag{3.14}$$

The prime indicates the derivative with respect to σ, the new time variable. Thus the 'classical' mechanics is now based on the Lagrangian

$$\mathcal{L} = \frac{ik}{(1-|\xi|^2)} (\xi^* \xi' - \xi \xi'^*) - \mathcal{H}(\xi, \xi^*). \tag{3.15}$$

In terms of τ and φ this becomes

$$\mathscr{L} = k(\cosh\tau)\dot{\varphi} - \mathscr{H}(\tau, \varphi) \tag{3.16}$$

where an overall total derivative $\dot{\varphi}$ has been dropped.

We now specify the functions U_1 and U_2. For the screened potential we have (case I)

$$U_1(r) = e^{-\lambda r} \tag{3.17a}$$

$$U_2(r) = 0 \tag{3.17b}$$

and for the charmonium potential we have (case II)

$$U_1(r) = 1 \tag{3.18a}$$

$$U_2(r) = \lambda r^2. \tag{3.18b}$$

Thus we have for case I

$$\Omega_I(E) = (2\mu)^{-1} e^{\theta}(K_0 + K_1) - E\, e^{-\theta}(K_0 - K_1) - \alpha \exp[-\lambda\, e^{-\theta}(K_0 - K_1)] \tag{3.19}$$

and for case II

$$\Omega_{II}(E) = (2\mu)^{-1} e^{\theta}(K_0 + K_1) - E\, e^{-\theta}(K_0 - K_1) - \alpha + \lambda\, e^{-2\theta}(K_0 - K_1)^2. \tag{3.20}$$

We must make a choice for the tilting angle θ. With $\theta = \ln[(-2E/\mu)^{1/2}]$ as before we have

$$\Omega_I^{(-)}(E) = (-2E/\mu)^{1/2} K_0 - \alpha \exp[-\lambda(-\mu/2E)^{1/2}(K_0 - K_1)] \tag{3.21}$$

and

$$\Omega_{II}^{(-)}(E) = (-2E/\mu)^{1/2} K_0 - \alpha + (-\mu/2E)\lambda(K_0 - K_1)^2. \tag{3.22}$$

This is the usual transformation for the bound-state case when $E < 0$. However, for the charmonium potential, the energy can also be positive. It proves convenient to choose the value of θ that would eliminate the compact generator K_0 from equation (2.7). Thus with $\theta = \ln[(2E/\mu)^{1/2}]$ we obtain, for the case $E > 0$,

$$\Omega_{II}^{(+)}(E) = (2E/\mu)^{1/2} K_1 - \alpha + (\mu/2E)(K_0 - K_1)^2. \tag{3.23}$$

We need the matrix elements of various operators such as K_0, K_1 and $(K_0 - K_1)^2$ in terms of the coherent states. From the results of previous work (Gerry *et al* 1983) we have

$$\langle \xi, k | K_0 | \xi, k \rangle = k \cosh\tau \tag{3.24}$$

$$\langle \xi, k | K_1 | \xi, k \rangle = -k \sinh\tau \cos\varphi. \tag{3.25}$$

Also, using these results we may show that

$$\begin{aligned} \langle r^2 \rangle &= \langle \xi, k | (K_0 - K_1)^2 | \xi, k \rangle \\ &= \langle r \rangle^2 + O(k) \end{aligned} \tag{3.26}$$

where $\langle r \rangle^2$ from equations (3.4) and (3.25) is of the order k^2. Since for large N k is large, we thus follow the standard large-N practice that

$$\begin{aligned} \langle r^2 \rangle &\approx \langle r \rangle^2 \\ &\approx k^2(\cosh\tau + \sinh\tau\cos\varphi)^2. \end{aligned} \tag{3.27}$$

For the Yukawa potential we must evaluate the quantity

$$\langle \xi, k| \exp[-\lambda(-\mu/2E)^{1/2}(K_0 - K_1)]|\xi, k\rangle \tag{3.28}$$

in the limit of large k. In fact, this may be treated as the matrix element of a finite group transformation of the parabolic type with an analytic continuation (Gerry and Laub 1984, 1985). However, we show (in the appendix) that in the large-k (or N) limit we may simply use

$$\langle \xi, k| \exp[-\lambda(-\mu/2E)^{1/2}(K_0 - K_1)]|\xi, k\rangle$$

$$\approx \exp[-\lambda(-\mu/2E)^{1/2}\langle \xi, k|K_0 - K_1|\xi, k\rangle] \tag{3.29}$$

$$\approx \exp[-\lambda(-\mu/2E)^{1/2}k(\cosh\tau + \sinh\tau\cos\varphi)]. \tag{3.30}$$

Thus using these results the energy relations are, since Ω is essentially a zero-energy Hamiltonian,

$$\mathscr{H}_{\mathrm{I}}^{(-)} = (-2E/\mu)^{1/2}k\cosh\tau - \alpha$$

$$\times \exp[-\lambda(-\mu/2E)^{1/2}k(\cosh\tau + \sinh\tau\cos\varphi)] = 0. \tag{3.31}$$

Similarly

$$\mathscr{H}_{\mathrm{II}}^{(-)} = (-2E/\mu)^{1/2}k\cosh\tau - \alpha + \lambda(-\mu/2E)k^2(\cosh\tau + \sinh\tau\cos\varphi)^2 = 0 \tag{3.32}$$

$$\mathscr{H}_{\mathrm{II}}^{(+)} = -(2E/\mu)^{1/2}k\sinh\tau\cos\varphi + \lambda(\mu/2E)k^2(\cosh\tau + \sinh\tau\cos\varphi)^2 = 0. \tag{3.33}$$

Now, from equation (3.16), we have the canonical momentum as

$$p = \partial\mathscr{L}/\partial\varphi = k\cosh\tau. \tag{3.34}$$

We therefore make the ansatz that it satisfies the Bohr-Sommerfeld quantisation rule

$$\oint p\,\mathrm{d}\varphi = 2\pi n \qquad n = 1, 2, 3, \ldots \tag{3.35}$$

or

$$\oint \cosh\tau\,\mathrm{d}\varphi = 2\pi n/k. \tag{3.36}$$

Thus $\cosh\tau$ is essentially a momentum and $1/k$ plays the role of Planck's constant $\hbar$. To check that this makes sense we consider the case where $\lambda = 0$, so that

$$\mathscr{H}^{(-)} = (-2E/\mu)^{1/2}k\cosh\tau - \alpha = 0 \tag{3.37}$$

$\cosh\tau = \alpha[k(-2E/\mu)^{1/2}]^{-1}$ and the phase integral rule give

$$2\pi\alpha/k(-2E/\mu)^{1/2} = 2\pi n/k \tag{3.38}$$

or

$$E_n = -\mu\alpha^2/2n^2 \tag{3.39}$$

the correct point Coulomb energy levels. We note that the orbit in polar phase space $(\cosh\tau, \varphi)$ is just a circle. On the other hand, when $\lambda = 0$ and for $E > 0$ we have, from equation (3.33),

$$\mathscr{H}^{(+)} = -(2E/\mu)^{1/2}k\sinh\tau\cos\varphi - \alpha = 0.$$

In this case the trajectory in phase space does not close and the Bohr-Sommerfeld quantisation rule cannot be applied, consistent with the fact that $E > 0$ states are not bound states.

4. Applications

In this section our phase integral approximation, equation (3.36), is applied to the Yukawa potential case ($\mu = 1$) with the energy constraint condition of equation (3.31), and to the Coulomb plus linear potential ($\mu = \frac{1}{2}$) with the energy constraints of equation (3.32) for $E < 0$ or (3.33) for $E > 0$.

The results for the Yukawa s states are in table 1. For comparison we have also included the exact results along with the usual JWKB calculations. For low λ values ours are in fairly close agreement with the other calculations but tend to be slightly on the high side. Nevertheless it is clear that our approximation method yields results indicating that large-N (or large-k) approximations have some validity as semiclassical approximations outside of the pseudospin and oscillator type of approximations considered previously. For $l \neq 0$, the orbits in phase space are quite complicated in that they do not encompass the origin (at $\tau = 0$). The phase integral is therefore much more complicated to calculate, although in principle it is possible to do so. Such calculations were performed in our previous work on anharmonic oscillators (Gerry *et al* 1983). In the present case, however, the maximum angular extensions are difficult to locate numerically.

Table 1. Energy eigenvalues for the Yukawa potential ($l = 0$ only).

λ	n	E (exact)	E (CS)	% error	E (JWKB)	% error
0.025	1	−0.475 4	−0.475 0	0.08	−0.475 4	
	2	−0.101 8	−0.101 6	0.20	−0.101 7	0.10
	3	−0.034 33	−0.034 21	0.35	−0.034 30	0.09
	4	−0.012 51	−0.012 4	0.80	−0.012 48	0.24
0.050	1	−0.451 8	−0.450 0	0.40	−0.451 7	0.02
	2	−0.081 77	−0.081 28	0.60	−0.081 67	0.12
	3	−0.019 35	−0.018 99	1.9	−0.019 28	0.36
	4	−0.003 09	−0.002 87	7.1	−0.003 05	1.3
0.10	1	−0.407 1	−0.399 8	1.8	−0.406 6	0.12
	2	−0.049 93	−0.048 36	3.1	−0.049 62	0.62
	3	−0.003 21	−0.002 46	23.0	−0.003 08	4.0
0.20	1	−0.326 8	−0.297 8	8.9	−0.325 1	0.52
	2	−0.012 11	−0.008 45	30.2	−0.001 147	5.3

For the Coulomb plus linear potential the results are in general agreement for the excited states (see table 2). We note that in this case the results tend to be lower whereas for the Yukawa potential the calculations give higher results than for the exact calculations. This tendency is also evident in the JWKB calculations though the errors are smaller.

5. Conclusions

In this paper we have applied our large-N phase integral approximation based on coherent states of SO(2, 1) (SU(1, 1)) to problems of the Coulomb type. This requires a different realisation of the Lie algebra and a different representation than was used for a similar application for the perturbed oscillator problems treated before (Gerry *et al* 1983).

Table 2. Energy eigenvalues for the Coulomb plus linear potential.

λ	E_{nl}	E (exact)	E (CS)	% error	E (JWKB)	% error
0.01	E_{10}	−0.221	−0.229	−3.6	−0.223	−0.90
	E_{30}	0.142	0.135	−4.9	0.141	−0.70
	E_{50}	0.286	0.279	−2.5	0.286	
	E_{32}	0.102	0.090	−12.0	0.102	
	E_{52}	0.251	0.241	−4.0	0.251	
	E_{72}	0.366	0.356	−2.7	0.366	
1	E_{10}	1.398	1.000	−28.5	1.379	−1.4
	E_{30}	5.033	4.756	−5.50	5.027	−0.12
	E_{50}	7.575	7.337	−3.14	7.571	−0.05
100	E_{10}	46.402	100.00	115.5	46.532	0.28
	E_{30}	116.74	110.12	−5.67	116.78	0.03
	E_{50}	169.50	163.83	−3.35	169.48	−0.01

It is fair to say that the present calculations do not have the accuracy of our previous work. There is no reason obvious to us for this behaviour but we speculate that it may be due to the fact that E enters the energy constraint equations in a very non-linear way, in contrast to the linear way it appears in the oscillator-type problems. Nevertheless, we do obtain results that have the same sort of behaviour as do the JWKB calculations, indicating that the large-N phase integral approximation is indeed a kind of semiclassical limit. We are currently searching for a way to obtain more accuracy for applications to the Coulomb-type problems.

Appendix

We wish to calculate

$$\langle \xi, k| \exp[-\lambda\beta(K_0 - K_1)]|\xi, k\rangle \qquad \beta = (-\mu/2E)^{1/2} \tag{A1}$$

in the limit of large k. Using the shorthand notation $\langle \hat{O}\rangle = \langle \xi, k|\hat{O}|\xi, k\rangle$ we have the following expansion:

$$\langle e^{-\lambda\beta r}\rangle = \left\langle \sum_{m=0}^{\infty} (-1)^m (\lambda\beta r)^m \right\rangle \tag{A2}$$

where $r = K_0 - K_1$. Now it can be shown that

$$\langle r\rangle = kf(\tau, \varphi) = kf(\xi) \tag{A3}$$

$$\langle r^m\rangle = k^m f^m(\xi) + \sum_{\nu=1}^{m-1} k^{m-\nu} g_{(\nu)}(\xi) \tag{A4}$$

where $f(\tau, \varphi) = f(\xi) = \cosh\tau + \sinh\tau\cos\varphi$ and $g_\nu(\xi)$ are functions of τ and φ we need not specify here. Thus the mth term of the expansion is

$$\lambda^m\beta^m\left(k^m f^m(\xi) + \sum_{\nu=1}^{m-1} k^{m-\nu} g_{(\nu)}(\xi)\right). \tag{A5}$$

We now rescale λ as λ/k so that we have

$$\lambda^m\beta^m\left(f^m(\xi) + \sum_{\nu=1}^{m-1} k^{-\nu} g_{(\nu)}(\xi)\right) \to \lambda^m\beta^m f^m(\xi). \tag{A6}$$

Restoring the original coupling constant, in the large-k limit we have

$$\langle e^{-\lambda\beta r}\rangle|_{k\to\infty} = \sum_{m=0}^{\infty} (-1)^m \lambda^m \beta^m \langle r\rangle^m$$
$$= \exp(-\lambda\beta\langle r\rangle). \tag{A7}$$

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